Gap solitons in quadratically nonlinear gratings: Beyond the cascading limit

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We consider pulse propagation in quadratically nonlinear gratings. Assuming that the phase mismatch between the fundamental and the second-harmonic frequencies δk is large, we present a perturbation method in δk^{-1}. In the well known cascading limit, terms to δk^{-1} are kept; here we keep terms to δk^{-2}, which leads to another type of coupled mode equations. Numerical calculation of the full equations support our theoretical results.

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It is well known that the group velocity of light propagation in a periodic structure can be much smaller than that in the uniform medium [1]. However, the dispersion introduced by the periodicity causes such pulses to broaden rapidly. This broadening can be counteracted if the structure is also nonlinear [2,3], as was demonstrated experimentally in fiber [4] and semiconductor geometries [5], all of which used a Kerr nonlinearity. Since the Kerr effect is a third order effect, and is thus quite weak, large optical powers were required; for example, in the fiber experiments the peak intensity was around 10 GW/cm^2, and was somewhat lower in the semiconductor geometry.

It is also well known that, in an appropriate limit, a quadratic nonlinearity leads to a nonlinear phase shift that is somewhat similar to that from the Kerr effect [6]. However, the quadratic effect has the advantage over a cubic effect that it can be much stronger. The starting point for studies of quadratically nonlinear periodic media is a set of four coupled mode equations. This work has shown that these media support pulselike solutions that can propagate at arbitrary velocities up to the speed of light in the medium [7–9]. In spite of this progress, general analytic solutions are not known, except for some limiting cases. In the first of these, the pulses are required to be wide so that a set of coupled nonlinear Schrödinger equations apply [7,9]. The second limit is the integrable case for stationary solitons in which the phase mismatch between the fundamental and the second harmonic is required to have a particular value [10]. The third limit is that of a large refractive index mismatch between the fundamental and the second harmonic frequencies; in this ''cascading limit,'' the quadratically nonlinear medium acts approximately as if it had a cubic nonlinearity, and the four coupled mode equations reduce to two [8,11,12].

The cascading limit is obtained when $\delta k \equiv k_2 - 2k_1 \rightarrow \infty$, where $k_{1,2}$ are the wave numbers at the fundamental and second-harmonic frequencies, respectively. It can thus be considered to be the first term in a $(\delta k)^{-1}$ expansion. An expansion in $(\delta k)^{-1}$ for soliton solutions in uniform media was considered by Buryak [13]. Higher order effects for solitons in the presence of a grating were studied by Conti *et al.* [14]. Retaining up to $O(\delta k^{-2})$, they obtained solutions that are either stationary or have a specified velocity. However, they do not obtain simple analytical expressions.

It thus seems that there is no complete generalization of coupled mode theory beyond the cascading limit, even up to $(\delta k)^{-2}$, and general analytic results have also not been found. Here we generalize the cascading limit, deriving *partial* differential equations for the forward and backward waves. We also give an analytical form for the soliton solutions with arbitrary velocity and detuning.

We start from the coupled mode equations that describe periodic systems with a $\chi^{(2)}$ nonlinearity [7–9]:

$$i\left(\frac{\partial}{\partial t} \pm \frac{\partial}{\partial z}\right)\mathcal{E}_{1\pm} + \kappa_1^{\pm}\mathcal{E}_{1\mp} + \Gamma(\mathcal{E}_{1\pm})^* \mathcal{E}_{2\pm} = 0, \tag{1}$$

$$i\left(\frac{\partial}{\partial t} \pm \sigma\frac{\partial}{\partial z}\right)\mathcal{E}_{2\pm} + \kappa_2^{\pm}\mathcal{E}_{2\mp} + \delta k \mathcal{E}_{2\pm} + \Gamma(\mathcal{E}_{1\pm})^2 = 0, \tag{2}$$

where $\mathcal{E}_{1,2\pm}$ are the envelopes of the forward $(+)$ and backward $(-)$ propagating modes at the fundamental (1) and second-harmonic (2) frequencies. Further, Γ is a real parameter that is proportional to $\chi^{(2)}$, $\kappa_{1,2} = \kappa_{1,2}^{+} = (\kappa_{1,2}^{-})^*$ are the grating strengths at the two frequencies, and δk was defined above. The group velocity of the fundamental and second-harmonic waves are normalized to 1 and σ, respectively. By suitable choosing the origin, κ_1 can be made a real positive number.

In Eqs. (1) and (2) all quantities are assumed dimensionless and normalized as

$$\mathcal{E}_{1\pm} \sim \frac{\partial}{\partial z}\mathcal{E}_{1\pm} \sim \frac{\partial}{\partial t}\mathcal{E}_{1\pm} \sim \kappa_1 \sim \kappa_2 \sim \Gamma \mathcal{E}_{2\pm} \sim O(1). \tag{3}$$

Here we are interested in large mismatches $\delta k \gg 1$, so that Eq. (2) is formally solved as an expansion of (δk^{-1})

$$\begin{pmatrix} \mathcal{E}_{2+} \\ \mathcal{E}_{2-} \end{pmatrix} = -\frac{\Gamma}{\delta k}\left\{ I - \frac{\boldsymbol{X}}{\delta k} + \frac{\boldsymbol{X}^2}{(\delta k)^2} - \cdots \right\} \begin{pmatrix} (\mathcal{E}_{1+})^2 \\ (\mathcal{E}_{1-})^2 \end{pmatrix}, \tag{4}$$

where the operator $\boldsymbol{X}$ is defined as

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$$\boldsymbol{X}=\begin{pmatrix} i\left(\frac{\partial}{\partial t}+\sigma\frac{\partial}{\partial z}\right) & \kappa_2^+ \\ \kappa_2^- & i\left(\frac{\partial}{\partial t}-\sigma\frac{\partial}{\partial z}\right)\end{pmatrix}. \tag{5}$$

The lowest order approximation, $\Gamma^{-1}\mathcal{E}_{2\pm}=-(\mathcal{E}_{1\pm})^2/\delta k$, corresponds to the cascading limit [11]. Note that according to Eq. (3) the above relation gives $\Gamma^2\sim\delta k$, and the nonlinear constant Γ should thus be large and $\mathcal{E}_{2\pm}$ should be small of $O(\delta k^{-1/2})$.

The second-order approximations for $\mathcal{E}_{2\pm}$ gives

$$\Gamma^{-1}\mathcal{E}_{2\pm}=-\frac{(\mathcal{E}_{1\pm})^2}{\delta k}+\frac{\kappa_2^{\pm}(\mathcal{E}_{1\mp})^2}{(\delta k)^2}-\frac{2\kappa_1\mathcal{E}_{1\pm}\mathcal{E}_{1\mp}}{(\delta k)^2}+\frac{2\Gamma^2}{(\delta k)^3}|\mathcal{E}_{1\pm}|^2(\mathcal{E}_{1\pm})^2\pm i\frac{\sigma-1}{(\delta k)^2}\frac{\partial}{\partial z}(\mathcal{E}_{1\pm})^2, \tag{6}$$

where we have eliminated time derivatives of $\mathcal{E}_{1\pm}$ by using Eq. (1). We refer to an analysis based on Eq. (6), as the *improved cascading* approximation. Note that in their analytic work Peschel *et al.* only retain the first and second terms of Eqs. (6) [8]. Substituting Eqs. (6) into Eqs. (1), we obtain a set of equations for the $\mathcal{E}_{1\pm}$ only

$$i\frac{\partial}{\partial t}\begin{pmatrix}\mathcal{E}_{1\pm}\\ \mathcal{E}_{1\pm}^*\end{pmatrix}=\left(1-\frac{2\Gamma^2}{(\delta k)^2}\left|\mathcal{E}_{1\pm}\right|^2\right)\begin{pmatrix}-\dfrac{\delta}{\delta\mathcal{E}_{1\pm}^*}\\ \dfrac{\delta}{\delta\mathcal{E}_{1\pm}}\end{pmatrix}H, \tag{7}$$

where

$$H=\int_{-\infty}^{+\infty}(\mathcal{H}_++\mathcal{H}_-)dz, \tag{8}$$

$$\mathcal{H}_{\pm}=(-1)^{\pm1}i\mathcal{E}_{1\pm}^*\frac{\partial\mathcal{E}_{1\pm}}{\partial z}\left(1+\frac{\Gamma^2\sigma}{(\delta k)^2}|\mathcal{E}_{1\pm}|^2\right)-\frac{\Gamma^2}{2\,\delta k}|\mathcal{E}_{1\pm}|^4+\kappa_1\mathcal{E}_{1\mp}\mathcal{E}_{1\pm}^*+\frac{\Gamma^2\kappa_2^{\pm}}{2(\delta k)^2}(\mathcal{E}_{1\mp}\mathcal{E}_{1\pm}^*)^2, \tag{9}$$

up to the order of interest. Other than the factor $(1-2\Gamma|\mathcal{E}_{1\pm}|^2/(\delta k)^2)$, Eq. (7) is of symplectic form in which H is an energylike integral. We can make it symplectic by a transformation of the dependent variable

$$\mathcal{E}_{1\pm}=E_{\pm}-\frac{\Gamma^2}{2(\delta k)^2}|E_{\pm}|^2E_{\pm}. \tag{10}$$

Applying the Poisson bracket $\{F,G\}_P$ with respect to $E_{\pm}$, we find that $\{\mathcal{E}_{1\pm}(x),\mathcal{E}_{1\pm}^*(y)\}_P\neq\{E_{\pm}(x),E_{\pm}^*(y)\}_P$. Transformation (10) is thus not canonical [15], and changes the symplectic structure. Using the new variable and neglecting $O(\delta k^{-2})$ quantities, we now obtain the coupled mode equations

$$i\left(\frac{\partial}{\partial t}\pm\frac{\partial}{\partial z}\right)E_{\pm}+\kappa_1E_{\mp}+\Gamma_0|E_{\pm}|^2E_{\pm}+\frac{\Gamma_1}{2}(|E_{\mp}|^2+2|E_{\pm}|^2)E_{\mp}+\frac{\Gamma_1}{2}(E_{\pm})^2E_{\mp}^*+\Gamma_2^{\pm}E_{\mp}^2E_{\pm}^*+\Gamma_3|E_{\pm}|^4E_{\pm}\pm i\Gamma_4|E_{\pm}|^2\frac{\partial E_{\pm}}{\partial z}=0, \tag{11}$$

where

$$\Gamma_0\equiv-\frac{\Gamma^2}{\delta k},\quad \Gamma_1\equiv-\frac{\Gamma^2}{(\delta k)^2}\kappa_1,\quad \Gamma_2^{\pm}\equiv\frac{\Gamma^2}{(\delta k)^2}\kappa_2^{\pm},$$

$$\Gamma_3\equiv\frac{3\Gamma^4}{(\delta k)^3},\quad \Gamma_4\equiv\frac{2\Gamma^2(\sigma-1)}{(\delta k)^2}. \tag{12}$$

Equations (11) constitute a homogeneous symplectic form [14] $i\partial E_{\pm}/\partial t=\pm\,\delta H/\delta E_{\pm}^*$. They are similar to the generalized coupled mode equations for deep (superstructure) gratings with a Kerr nonlinearity [16,17], except that our result has quintic and nonlinear derivative terms, and does not have cross phase modulation terms.

We find traveling solutions to Eq. (11) of the form

$$E_{\pm}=\Delta^{\mp1/2}[1+\Delta_{\pm}(\zeta)][F(\zeta)]^{1/2}e^{i[\theta_{\pm}(\zeta)-\Omega t]}, \tag{13}$$

where $\zeta=z-Vt$, Δ is a positive constant, Ω is the detuning parameter, and the $\Delta_{\pm}$ are assumed small, of $O(\delta k^{-1})$. Note that the envelopes of the forward and backward waves differ in shape if the $\Delta_{\pm}$ are different. Note also that since the two complex equations (13) contain five unknown real functions, we may require an additional condition, which we here impose on $\Delta_++\Delta_-$.

Substituting Eq. (13) into Eqs. (11) we have

$$\Delta=\sqrt{(1-V)/(1+V)}, \tag{14}$$

$$\Delta_+(\zeta)-\Delta_-(\zeta)=-\gamma^3V\Gamma_4F(\zeta), \tag{15}$$

where $\gamma=(1-V^2)^{-1/2}$. If we choose,

$$\Delta_+(\zeta)+\Delta_-(\zeta)=-\frac{\gamma}{2}(2\gamma^2-1)\Gamma_4F(\zeta), \tag{16}$$

we obtain an ODE system for F and $\phi\equiv\theta_+-\theta_-$

$$\frac{d\phi}{d\zeta}=\frac{\partial I}{\partial F},\quad \frac{dF}{d\zeta}=-\frac{\partial I}{\partial\phi}, \tag{17}$$

with the integral $I(F,\phi)$

$$\begin{aligned}I(F,\phi)={}&2\gamma^2\Omega F+2\gamma\kappa_1F\cos\phi+\{\gamma(2\gamma^2-1)\Gamma_0\\&-\gamma^3(4\gamma^2-3)\Omega\Gamma_4\}F^2+\{2\gamma^2\Gamma_1-2\gamma^2\\&\times(2\gamma^2-1)\kappa_1\Gamma_4\}F^2\cos\phi+\gamma|\Gamma_2|F^2\cos(2\phi-\beta)\\&+\left\{\frac{2}{3}\Gamma_3(4\gamma^4-3\gamma^2)-\gamma^2(8\gamma^4-8\gamma^2+1)\right.\\&\left.\times\Gamma_4\Gamma_0\right\}F^3,\end{aligned}$$

where $\beta=\arg(\kappa_2)$.

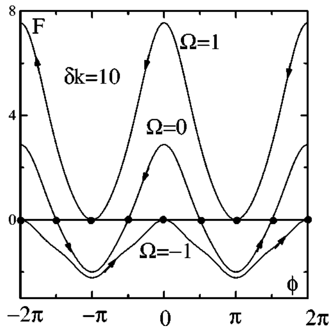

FIG. 1. Phase flows in F-ϕ space for $\kappa_1=\kappa_2=\sigma=1$, $\Gamma=3$, $V=0$, and $\delta k=10$. Solitary wave solutions are allowed only in the gap ($|\Omega|<1$). $F<0$ is unphysical as seen in Eq. (13). Solid dots indicate fixed points.

Here we search for localized waves, thus $I(F,\phi)=0$, and we obtain a relation between F and ϕ that reads

$$F[\phi]=f[\phi]\{1+a_1+a_2\cos\phi+a_3\cos(2\phi-\beta)\}, \quad (18)$$

where

$$f[\phi]\equiv-\frac{2(\gamma\Omega+\kappa_1\cos\phi)}{(2\gamma^2-1)\Gamma_0}, \quad (19)$$

$$a_1\equiv\frac{4(4\gamma^4-3\gamma^2)}{3(2\gamma^2-1)^2}\frac{\Gamma_3\Omega}{\Gamma_0^2}-\frac{\gamma^2(8\gamma^4-6\gamma^2-1)}{(2\gamma^2-1)^2}\frac{\Gamma_4\Omega}{\Gamma_0}, \quad (20)$$

$$a_2\equiv-\frac{2\gamma\Gamma_1}{(2\gamma^2-1)\Gamma_0}+\frac{4(4\gamma^3-3\gamma)}{3(2\gamma^2-1)^2}\frac{\kappa_1\Gamma_3}{\Gamma_0^2}-\frac{\gamma^2(12\gamma^4-12\gamma^2+1)}{(2\gamma^2-1)^2}\frac{\kappa_1\Gamma_4}{\Gamma_0}, \quad (21)$$

$$a_3\equiv-\frac{|\Gamma_2|}{(2\gamma^2-1)\Gamma_0}. \quad (22)$$

Equation (18) gives the phase flows to $O(\delta k^{-1})$ in F-ϕ space; typical flows for stationary solutions are shown in Fig. 1, for $\delta k=10$ and $\kappa_1=\kappa_2=\sigma=1$, $\Gamma=3$, $\beta=0$.

Applying the case of a solitary wave with $I=0$ to the ODE for ϕ, we have

$$\frac{d\phi}{2\gamma^2\Omega+2\gamma\kappa_1\cos\phi}=-\lambda\,d\phi-d\zeta, \quad (23)$$

$$\lambda\equiv\frac{2(4\gamma^2-3)\Gamma_3}{3(2\gamma^2-1)^2\Gamma_0^2}-\frac{8\gamma^4-8\gamma^2+1}{(2\gamma^2-1)^2}\frac{\Gamma_4}{\Gamma_0}\sim O\left(\frac{1}{\delta k}\right). \quad (24)$$

In the cascading limit $\lambda\to0$, $\phi(\zeta)=\phi_0(\zeta)$ is given by [12,17]

$$\phi_0(\zeta)\equiv-2\arctan\left[\sqrt{\frac{\kappa_1+\gamma\Omega}{\kappa_1-\gamma\Omega}}\tanh^{\mp1}(\xi/2)\right], \quad (25)$$

$$\xi\equiv2\{\sqrt{\kappa_1^2-\gamma^2\Omega^2}\,\gamma(\zeta-\zeta_0)\}, \quad (26)$$

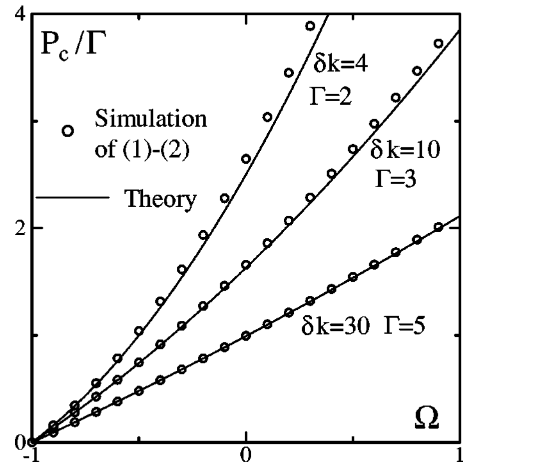

FIG. 2. Normalized soliton center power P_c/Γ. Solid lines are the analytic result (28). Circles indicates numerical values from Eqs. (1) and (2). Parameters are set as $\kappa_1,\kappa_2,\sigma=1$, and $V=0$. Region $|\Omega|<1$ corresponds to the photonic band gap.

where ζ_0 is a constant. Comparing Eq. (23) with the differential equation for ϕ_0: $d\phi_0/(2\gamma^2\Omega+2\gamma\kappa_1\cos\phi_0)=-d\zeta$, we obtain $\phi(\zeta)$ in implicit form $\phi(\zeta)=\phi_0(\zeta+\lambda\phi)$. Noting that λ is small, we may replace ϕ on the right-hand side by ϕ_0,

$$\phi(\zeta)=\phi_0(\zeta+\lambda\phi_0(\zeta)). \quad (27)$$

Function $\phi_0(\zeta)$ is already given in Eq. (25), and we have thus an explicit expression for the phase $\phi(\zeta)$ up to $O(\delta k^{-1})$. We can also obtain the "soliton amplitude" $F(\zeta)$ by substituting Eq. (27) into Eq. (18).

Applying transformation (10) and noting Eq. (12), we directly obtain the total power $P=|\mathcal{E}_{1+}|^2+|\mathcal{E}_{1-}|^2$ in an analytic form:

$$P(\zeta)=2\gamma f(\zeta)\left(1+b_1\frac{\Omega}{\delta k}+b_2\frac{\kappa_1\cos\phi(\zeta)}{\delta k}+\frac{|\kappa_2|\cos(2\phi(\zeta)-\beta)}{(2\gamma^2-1)\delta k}\right), \quad (28)$$

where $f(\zeta)=f[\phi(\zeta)]$ and

$$b_1=\frac{2(4\gamma^4-2\gamma^2-1)}{(2\gamma^2-1)^2}+\frac{8\gamma^2(\gamma^2-1)}{(2\gamma^2-1)^2}(\sigma-1), \quad (29)$$

$$b_2=\frac{4\gamma^4-2\gamma^2-2}{(2\gamma^2-1)^2\gamma}\{1+2\gamma^2(\sigma-1)\}. \quad (30)$$

Thus we can obtain an analytic expression for the center value P_c. In Fig. 2 we compare the analytic result for P_c with the numerically obtained value from the original system (1) and (2), for different values of δk and Ω. The numerical results were obtained assuming the envelopes to be of the form $\mathcal{E}_{1,2\pm}=g_{1,2\pm}(\zeta)e^{-i\Omega t}$ which reduces Eqs. (1) and (2) to an ODE system for the $g_{1,2\pm}(\zeta)$.

Clearly for large δk ($=30$), the agreement is good everywhere in the photonic band gap. It is surprising that even if

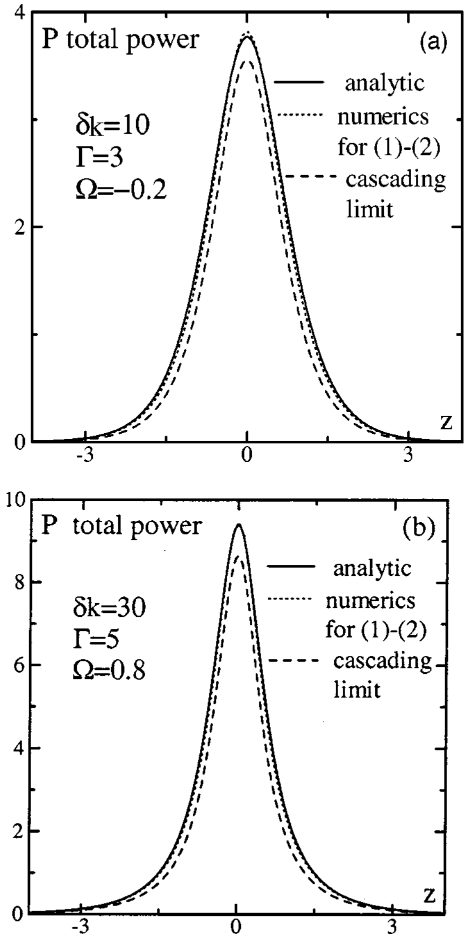

FIG. 3. Comparison of soliton profiles obtained from numerically solving Eqs. (1) and (2) (short dashed lines), in the cascading limit (long dashed lines), and in the improved cascading limit (solid lines). The parameters not given in the figure are $\kappa_1,\kappa_2,\sigma=1$, and $V=0$.

$\delta k=4$ [in a sense of $O(1)$], Eq. (28) is effective particularly for small Ω. For higher powers, the deviations between the two results are more obvious, though remain modest. Even though P_c/Γ should be small enough to apply our theory, Fig. 2 shows good agreement even if P_c/Γ is of order unity.

In Fig. 3 we compare the complete analytical results from Eq. (28) with numerical results from the original system (1) and (2), and results in the cascading limit. Equations (28) are clearly superior to the cascading results, and are almost indistinguishable from the exact results.

Our procedure can certainly be generalized to include higher orders of $(\delta k)^{-1}$. Actually using Eqs. (1) and (2) we can solve for the $\mathcal{E}_{2\pm}$ using the matrices $\mathbf{Y}^{(j)}$

$$\Gamma^{-1}\begin{pmatrix}\mathcal{E}_{2+}\\ \mathcal{E}_{2-}\end{pmatrix} = -\frac{1}{\delta k}\sum_{j=0}^{\infty}\frac{\mathbf{Y}^{(j)}}{(\delta k)^{j}}\begin{pmatrix}\mathcal{E}_{1+}^{2}\\ \mathcal{E}_{1-}^{2}\end{pmatrix}, \tag{31}$$

where $\mathbf{Y}^{(0)}$ is the unit matrix and $\mathbf{Y}^{(j)}$ are determined sequentially by

$$\mathbf{Y}^{(j)} = -\begin{pmatrix} i(\sigma-1)\dfrac{\partial}{\partial z} & \kappa_2^{+}-\dfrac{2\kappa_1\mathcal{E}_{1+}}{\mathcal{E}_{1-}} \\ \kappa_2^{-}-\dfrac{2\kappa_1\mathcal{E}_{1-}}{\mathcal{E}_{1+}} & -i(\sigma-1)\dfrac{\partial}{\partial z}\end{pmatrix}\mathbf{Y}^{(j-1)} - \frac{2\Gamma^2}{\delta k}\begin{pmatrix}|\mathcal{E}_{1+}|^2 & 0\\ 0 & |\mathcal{E}_{1-}|^2\end{pmatrix}\sum_{l=1}^{j}\mathbf{Y}^{(j-l)}\mathbf{Y}^{(l-1)}. \tag{32}$$

Substituting the above $\mathcal{E}_{2\pm}$ into Eq. (1), we obtain generalized coupled mode equations for $\mathcal{E}_{1\pm}$ up to an arbitrary order, but solving them seems to be increasingly tedious; we therefore do not discuss it here.

Numerical methods for solving to Eqs. (1) and (2) are well known [8–10,12,14]. Nonetheless, any nontrivial analytic result is of interest as it can point to trends that are difficult to unravel numerically. For example, Eq. (28) shows how the soliton amplitude depends on κ_2; this subtle dependence would be difficult to extract numerically.

In conclusion starting from the general coupled mode equations describing a periodic medium with a $\chi^{(2)}$ nonlinearity, we present an improvement to the usual cascading approximation. Though well studied, the cascading limit is somewhat crude and corresponds to treating the quadratic nonlinearity as a Kerr effect. We find that the fields approximately satisfy Eqs. (11), and, at this level, the cascaded nonlinearity can thus not be described by a cubic effect only. We give analytic expressions for the soliton solutions that agree well with numerical solutions of the full system. Though we assume a large mismatch, our results appear to be reliable for $O(1)$ mismatch if the detuning is close to the lower gap edge. Our method can also be generalized to higher orders.

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